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General expressions for the elastic displacement fields induced by dislocations in quasicrystals

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Abstract. The set of partial differential equations satisfied by the phonon and phason displacement fields $\boldsymbol{u}$ and $\boldsymbol{w}$ in quasicrystals has been solved by means of Fourier transform and eigenstrain methods, and general expressions of the elastic displacement fields induced by dislocations in quasicrystals have been given in terms of the Green function. The elastic Green tensor functions for every kind of quasicrystal are discussed in detail. Finally, as an example, the displacement fields induced by a straight dislocation line along the periodic tenfold axis of decagonal quasicrystals (three-dimensional) are calculated.

1. Introduction

According to the generalized elasticity theory of quasicrystals suggested by us [1], the system of inhomogeneous partial differential equations satisfied by the phonon and phason displacement fields, $\boldsymbol{u}$ and $\boldsymbol{w}$, are as follows:

$$
\begin{aligned}
&C_{ijkl}\partial_j\partial_l u_k + R_{ijkl}\partial_j\partial_l w_k + f_i = 0\\
&R_{klij}\partial_j\partial_l u_k + K_{ijkl}\partial_j\partial_l w_k + g_i = 0.
\end{aligned}
\tag{1}
$$

In equation (1), the definitions of C_{ijkl} and f_i are the same as in the classical elasticity theory, the components of the fourth-order tensor K_{ijkl} are called the second-order elastic constants of the phason field, R_{ijkl} are the elastic constants associated with the phonon–phason coupling, and g_i is a generalized body force in mathematical (complementary) space $P_\perp$ [1].

In the absence of the phason field $\boldsymbol{w}$, the system of equation (1) will be reduced to that for the usual crystals, which can be solved by the Green function method. Moreover, this method, as has been proved, is a powerful tool for solving problems in the continuum theory of defects, such as dislocations and disclinations. As we know, the systematic mature theory on elastic models of defects in crystals was established by the Green tensor function method and other methods about 20 years ago [2–4].

On the other hand, during the past few years some defects such as dislocations and stacking faults have also been observed experimentally in quasicrystals. Therefore, how to express the elastic field induced by these defects is an interesting problem in both theory and experiment. It is clear that this problem, considered here, is more difficult than that in crystals. In 1987, remarkable progress was made in this field by De *et al* [5], who derived a *system of inhomogeneous elastic equilibrium equations of the planar pentagonal structure* by the usual Euler–Lagrange procedure, and found the expressions of the displacement fields induced by a dislocation point in such a structure, in terms of the Green tensor function.

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In order to find a general expression applicable to all quasicrystals in which dislocations have been observed experimentally, it would be best to solve the elastic equilibrium equations (1) directly. In this paper, we will employ the Fourier transform method and another eigenstrain method used to treat the micromechanics of defects in solids [6], to solve the system of equations (1), and give the general expressions of the solution in terms of the elastic Green tensor function (in section 2). In section 3, we will give explicit algebraic expressions of the Green tensors of some quasicrystals when possible. In section 4 an application to the displacement field induced by a straight dislocation line in a decagonal quasicrystal, which is a three-dimensional (3D) body, will be given. Finally some problems are discussed in section 5.

2. General expressions of the solution in terms of the Green tensor function

By the definitions of the eigenstrain and eigenstress [6], we can assume that a subdomain V' in a body V is subjected to a given eigenstrain induced by phase transformations, plastic deformations or other kinds of non-elastic strains. It follows that an eigenstress field, i.e. a self-equilibrated internal stress field caused by the eigenstrain in V', will occur in the body V. From the nature of the case, there will be an elastic strain field responding to it in V. According to the elastic theory, the eigenstress is related to the elastic strain field by the generalized Hooke's law.

Here, we take two sets of orthogonal coordinate systems in the physical space $P_\parallel$ and in the mathematical space $P_\perp$, respectively, as we do in the projection method. Consequently, the phonon and phason strain fields, $E_{ij}(\boldsymbol{x})$ and $w_{ij}(\boldsymbol{x})$, are defined as follows:

$$E_{ij} = \tfrac{1}{2}(\partial_j u_i + \partial_i u_j) \qquad w_{ij} = \partial_j w_i \tag{2}$$

where $\boldsymbol{u}$ and $\boldsymbol{w}$ are the phonon and phason variables, respectively. All of $\boldsymbol{u}$, $\boldsymbol{w}$, E_{ij} and w_{ij} are functions of the position vector $\boldsymbol{x}$ (x_1, x_2, x_3) in $P_\parallel$. It should be noted that in $w_{ij} = \partial_j w_i$, $w_i s$ are components in $P_\perp$, but $\partial_j \equiv \partial/\partial x_j$ are derivatives with respect to coordinate variables x_j in $P_\parallel$. The eigenstrain fields denoted by $*$ induced by defects and the elastic strain fields denoted by $'$ in quasicrystals must consist of two parts: E^*_{ij}, E'_{ij} associated with the phonon variable and w^*_{ij}, w'_{ij} associated with the phason variable. The latter are absent in the usual crystals. Therefore, the actual strain fields E_{ij} and w_{ij} in V are the sum of the eigenstrain and elastic strain fields:

$$E_{ij} = E^*_{ij} + E'_{ij} \qquad w_{ij} = w^*_{ij} + w'_{ij}. \tag{3}$$

Moreover, the related internal stress fields are also divided into two parts: T_{ij} and H_{ij}. Substituting equation (3) into the generalized Hooke's law [1], we can easily obtain the following expressions:

$$\begin{aligned} T_{ij} &= C_{ijkl}E_{kl} + R_{ijkl}w_{kl} - C_{ijkl}E^*_{kl} - R_{ijkl}w^*_{kl} \\ H_{ij} &= R_{klij}E_{kl} + K_{ijkl}w_{kl} - R_{klij}E^*_{kl} - K_{ijkl}w^*_{kl}. \end{aligned} \tag{4}$$

According to the eigenstrain method, we assume that a quasicrystal is an infinite, homogeneous and free body. Thus, we have $f_i = g_i = 0$ in equation (1), and the boundary conditions $T_{ij}n_j = H_{ij}n_j = 0$ when $\boldsymbol{x} \to \infty$.

Substituting equation (4) into the static equilibrium equations [1],

$$\partial_j T_{ij} = 0 \qquad \partial_j H_{ij} = 0 \tag{5}$$

we have

$$\begin{aligned} C_{ijkl}\partial_j\partial_l u_k + R_{ijkl}\partial_j\partial_l w_k &= C_{ijkl}\partial_j E^*_{kl} + R_{ijkl}\partial_j w^*_{kl} \\ R_{klij}\partial_j\partial_l u_k + K_{ijkl}\partial_j\partial_l w_k &= R_{klij}\partial_j E^*_{kl} + K_{ijkl}\partial_j w^*_{kl}. \end{aligned} \tag{6}$$

By comparing equation (6) with equation (1) one can see that the contribution of E^*_{kl} and w^*_{kl} to the equilibrium equations is similar to that of two body forces X_i and Y_i:

$$\begin{aligned} X_i &= -C_{ijkl}\partial_j E^*_{kl} - R_{ijkl}\partial_j w^*_{kl} \\ Y_i &= -R_{klij}\partial_j E^*_{kl} - K_{ijkl}\partial_j w^*_{kl}. \end{aligned} \tag{7}$$

The subsequent procedure is to solve equations (6). According to the definition of the Fourier integral and Fourier transform for a function $f(x)$, we have

$$\begin{aligned} f(\boldsymbol{x}) &= \int_{-\infty}^{\infty} f(\boldsymbol{k}) \exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x})\,\mathrm{d}\boldsymbol{k} \\ f(\boldsymbol{k}) &= \frac{1}{8\pi^3}\int_{-\infty}^{\infty} f(\boldsymbol{x}) \exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \end{aligned} \tag{8}$$

with $\mathrm{d}\boldsymbol{k} = \mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}k_3$ and $\mathrm{d}\boldsymbol{x} = \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3$.

Carrying out Fourier transformation for equations (6), term by term, we can find the system of equations:

$$\begin{aligned} M_{ik}\bar{u}_k + R_{ik}\bar{w}_k &= \bar{X}_i \\ R^{\mathrm{T}}_{ik}\bar{u}_k + N_{ik}\bar{w}_k &= \bar{Y}_i \end{aligned} \tag{9}$$

satisfied by the Fourier transform $\bar{u}_i(\boldsymbol{k})$ and $\bar{w}_i(\boldsymbol{k})$, where

$$\begin{aligned} M_{ik} &= C_{ijkl}k_jk_l \qquad & R_{ik} &= R_{ijkl}k_jk_l \\ R^{\mathrm{T}}_{ik} &= R_{klij}k_jk_l \qquad & N_{ik} &= K_{ijkl}k_jk_l \end{aligned} \tag{10}$$

and

$$\begin{aligned} \bar{X}_i &= -\mathrm{i}C_{ijkl}k_j\bar{E}^*_{kl} - \mathrm{i}R_{ijkl}k_j\bar{w}^*_{kl} \\ \bar{Y}_i &= -\mathrm{i}R_{klij}k_j\bar{E}^*_{kl} - \mathrm{i}K_{ijkl}k_j\bar{w}_{kl}. \end{aligned} \tag{11}$$

For convenience of the next derivation, it is better to change equations (9) to a general matrix form in the following way. First, we define a 6×6 symmetrical matrix $A^{\alpha\beta}$, which consists of four matrices of coefficients M_{ik}, R_{ik}, R^{T}_{ik} and N_{ik} on the left of equations (9):

$$A^{\alpha\beta} = \delta^\alpha_i(\delta^\beta_j M_{ij} + \delta^{\beta-3}_j R_{ij}) + \delta^{\alpha-3}_i(\delta^\beta_j R_{ji} + \delta^{\beta-3}_j N_{ij}). \tag{12}$$

Next, we define two pairs of 6D vectors, V^α, $\bar{V}^\alpha$ and Z^α, $\bar{Z}^\alpha$:

$$V^\alpha(\boldsymbol{x}) = \delta^\alpha_i u_i(\boldsymbol{x}) + \delta^{\alpha-3}_i w_i(\boldsymbol{x}) \qquad \bar{V}_\alpha(\boldsymbol{k}) = \delta^\alpha_i \bar{u}_i(\boldsymbol{k}) + \delta^{\alpha-3}_i \bar{w}_i(\boldsymbol{k}) \tag{13}$$

$$Z^\alpha(\boldsymbol{x}') = \delta^\alpha_i X_i(\boldsymbol{x}') + \delta^{\alpha-3}_i Y_i(\boldsymbol{x}') \qquad \bar{Z}^\alpha(\boldsymbol{k}) = \delta^\alpha_i \bar{X}_i(\boldsymbol{k}) + \delta^{\alpha-3}_i \bar{Y}_i(\boldsymbol{k}) \tag{14}$$

where

$$\delta_i^\alpha = \begin{cases} 1 & \alpha = i \\ 0 & \alpha \neq i \end{cases} \qquad \delta_i^{\alpha-3} = \begin{cases} 1 & \alpha - 3 = i \\ 0 & \alpha - 3 \neq i. \end{cases} \tag{15}$$

From now on we use indices $\alpha, \beta, \ldots (= 1, 2, 3, \ldots, 6)$ for superscripts and indices $i, j, \ldots (= 1, 2, 3)$ for subscripts. After these procedures, equations (9) can be expressed in the form of standard inhomogeneous algebraic equations:

$$A^{\alpha\beta}(k)\bar{V}^\beta(k) = \bar{Z}^\alpha(k). \tag{16}$$

The solutions of the algebraic equations can be expressed as:

$$\bar{V}^\alpha(k) = \bar{G}^{\alpha\beta}(k)\bar{Z}^\beta(k) \tag{17}$$

with

$$\bar{G}^{\alpha\beta}(k) = [A^{\alpha\beta}(k)]^{-1} = B^{\alpha\beta}(k)/D(k) \tag{18}$$

where $D(k)$ is the determinant of the matrix $A^{\alpha\beta}$ and $B^{\alpha\beta}(k)$ are the algebraic complements of $A^{\alpha\beta}$, and

$$\bar{Z}^\beta(k) = \frac{1}{8\pi^3}\int_{-\infty}^{\infty} Z^\beta(x')\exp(-\mathrm{i}k\cdot x')\,\mathrm{d}x'. \tag{19}$$

According to equations (8) and (19), we immediately obtain

$$V^\alpha(x) = \int_{-\infty}^{\infty} \bar{V}^\alpha(k)\exp(\mathrm{i}k\cdot x)\,\mathrm{d}k = \frac{1}{8\pi^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \bar{G}^{\alpha\beta}(k)Z^\beta(x')\exp[\mathrm{i}k\cdot(x-x')]\,\mathrm{d}k\,\mathrm{d}x'. \tag{20}$$

Usually, we would like to express the solutions of u and w in terms of Green functions $G^{\alpha\beta}(x - x')$ as

$$V^\alpha(x) = \int_{-\infty}^{\infty} G^{\alpha\beta}(x - x')Z^\beta(x')\,\mathrm{d}x'. \tag{21}$$

Comparing equations (21) and (20), we have

$$G^{\alpha\beta}(x - x') = \frac{1}{8\pi^3}\int_{-\infty}^{\infty} \bar{G}^{\alpha\beta}(k)\exp[\mathrm{i}k\cdot(x - x')]\,\mathrm{d}k \tag{22}$$

or

$$G^{\alpha\beta}(x - x') = \frac{1}{8\pi^3}\int_{-\infty}^{\infty} \frac{B^{\alpha\beta}(k)}{D(k)}\exp[\mathrm{i}k\cdot(x - x')]\,\mathrm{d}k. \tag{23}$$

Substituting equations (14) and (7) into (21), we finally obtain the solutions of V^α ($V^\alpha = u_i$ for $\alpha = 1, 2, 3$ and $V^\alpha = w_i$ for $\alpha = 4, 5, 6$) as

$$\begin{aligned} V^\alpha(x) = &-\int_{-\infty}^{\infty} G^{\alpha\beta}(x - x')(\delta_i^\beta C_{ijkl} + \delta_i^{\beta-3}R_{klij})\partial_{j'}E^*_{kl}(x')\,\mathrm{d}x' \\ &-\int_{-\infty}^{\infty} G^{\alpha\beta}(x - x')(\delta_i^\beta R_{ijkl} + \delta_i^{\beta-3}K_{ijkl})\partial_{j'}w^*_{kl}(x')\,\mathrm{d}x' \end{aligned} \tag{24}$$

or

$$V^{\alpha}(\boldsymbol{x}) = -\int_{-\infty}^{\infty} \partial_j G^{\alpha\beta}(\boldsymbol{x}-\boldsymbol{x}')(\delta_i^{\beta} C_{ijkl} + \delta_i^{\beta-3} R_{klij}) E_{kl}^{*}(\boldsymbol{x}')\,\mathrm{d}\boldsymbol{x}'$$
$$- \int_{-\infty}^{\infty} \partial_j G^{\alpha\beta}(\boldsymbol{x}-\boldsymbol{x}')(\partial_i^{\beta} R_{ijkl} + \delta_i^{\beta-3} K_{ijkl}) w_{kl}^{*}(\boldsymbol{x}')\,\mathrm{d}\boldsymbol{x}' \qquad (25)$$

where $\partial_{j'} = \partial/\partial x_{j'}$ in equation (24), $\partial_j = \partial/\partial x_j$ and $\partial_j = -\partial_{j'}$. Then the corresponding strain fields E_{ij}, w_{ij} and stress fields T_{ij}, H_{ij} can easily be obtained from equations (2) and the generalized Hooke's law [1], respectively.

In order to understand the physical meaning of the Green tensor $G^{\alpha\beta}(\boldsymbol{x}-\boldsymbol{x}')$, it is necessary to point out the difference of the subscripts i, j, k, l in C_{ijkl}, K_{ijkl} and R_{ijkl}. According to the transformation characteristic of the elastic coefficient tensor under symmetric operations of quasicrystals, subscripts i, j, k, l in C_{ijkl}, j, l in K_{ijkl} and i, j, l in R_{ijkl} denote the coordinate components in the physical space ($P_{\parallel}$), and i, k in K_{ijkl} and k in R_{ijkl} denote the coordinate components in the mathematical space ($P_{\perp}$) [7].

It is clear from equation (7) that the body forces X_i and Y_i both act at the point x_i' in $P_{\parallel}$, but the directions of X_i and Y_i are directed along different axes. The former is in $P_{\parallel}$ and the latter in $P_{\perp}$.

Therefore, equations (21) and (14) show that $G^{\alpha\beta}(\boldsymbol{x}-\boldsymbol{x}')$ represents the components of the phonon (when $\alpha = 1, 2, 3$) and phason (when $\alpha = 4, 5, 6$) displacements at the point $\boldsymbol{x}$ in $P_{\parallel}$. This displacement is produced by a unit point body force along the β direction acting at the point x_i' in $P_{\parallel}$. This body force is in $P_{\parallel}$ space when $\beta = 1, 2, 3$ or $P_{\perp}$ space when $\beta = 4, 5, 6$.

3. Explicit expressions of Green tensors of quasicrystals

In order to obtain the elastic field induced by defects in quasicrystals, the crux of the matter is to calculate the Green tensor $G^{\alpha\beta}(\boldsymbol{x})$. Clearly, when the matrices C_{ijkl}, K_{ijkl} and R_{ijkl} are given, $G^{\alpha\beta}(\boldsymbol{x})$ *can be, in principle, calculated by equations (10), (12), (18) and (22) or* (23). However, it can be seen from equations (18) and (22) or (23) that the calculations of the Green tensor involve Fourier integrals of $B^{\alpha\beta}(\boldsymbol{k})/D(\boldsymbol{k})$ where the denominator $D(\boldsymbol{k})$ is a high-order polynomial from 4 to 12, much higher than that in conventional crystals. Hence the practical calculation must be more difficult than that in usual crystals. In the following, we will discuss these Green tensors for various quasicrystals and derive explicit algebraic expressions of Green tensors for some quasicrystals when possible.

3.1. The Green tensors of planar quasicrystals (2D)

In this case, the Fourier integral (23) takes the following form:

$$G^{\alpha\beta}(\boldsymbol{x}-\boldsymbol{x}') = \frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \bar{G}^{\alpha\beta}(\boldsymbol{k}) \exp\{\mathrm{i}[k_1(x_1-x_1') + k_2(x_2-x_2')]\}\,\mathrm{d}k\,\mathrm{d}k_2 \qquad (26)$$

where $\alpha, \beta = 1, 2, 3, 4$ and $i, j, k, l = 1, 2$.

3.1.1. Planar quasicrystals of point group 5. For a planar pentagonal quasicrystal of point group 5 there are six independent elastic constants:

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{27}$$

$$R_{ijkl} = R_1(\delta_{i1} - \delta_{i2})(\delta_{ij}\delta_{kl} - \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + R_2[(1-\delta_{ij})\delta_{kl} + \delta_{ij}(\delta_{i1} - \delta_{i2})(\delta_{k1}\delta_{l2} - \delta_{k2}\delta_{l1})] \tag{28}$$

$$K_{ijkl} = K_1\delta_{ik}\delta_{jl} + K_2(\delta_{ij}\delta_{kl} - \delta_{il}\delta_{jk}). \tag{29}$$

Substituting equations (27)–(29) into (12), we obtain a 4×4 matrix:

$$[A^{\alpha\beta}] = \begin{bmatrix} (\lambda+\mu)k_1 + \mu k^2 & (\lambda+\mu)k_1k_2 & R_1(k_1^2 - k_2^2) + 2R_2k_1k_2 & 2R_1k_1k_2 - R_2(k_1^2 - k_2^2) \\ (\lambda+\mu)k_1k_2 & (\lambda+\mu)k_2^2 + \mu k^2 & R_2(k_1^2 - k_2^2) - 2R_1k_1k_2 & 2R_2k_1k_2 + R_1(k_1^2 - k_2^2) \\ R_1(k_1^2 - k_2^2) + 2R_2k_1k_2 & R_2(k_1^2 - k_2^2) - 2R_1k_1k_2 & K_1k^2 & 0 \\ 2R_1k_1k_2 - R_2(k_1^2 - k_2^2) & R_1(k_1^2 - k_2^2) + 2R_2k_1k_2 & 0 & K_1k^2 \end{bmatrix}. \tag{30}$$

After calculating the inverse $\bar{G}^{\alpha\beta}(k) = [A^{\alpha\beta}(k)]^{-1}$ from equation (30), and carrying out the Fourier integrals (26), we obtain explicit expressions of the Green tensor $G^{\alpha\beta}(\boldsymbol{x})$ for the planar quasicrystal of point group 5 as follows:

$$G^{11} = -\frac{K_1}{2\pi c_2}\ln r + \frac{K_1^2(\lambda+\mu)}{4\pi c_1 c_2}\left(\frac{x_1^2}{r^2} - \ln r\right) \tag{31a}$$

$$G^{22} = -\frac{K_1}{2\pi c_2}\ln r + \frac{K_1^2(\lambda+\mu)}{4\pi c_1 c_2}\left(\frac{x_2^2}{r^2} - \ln r\right) \tag{31b}$$

$$G^{33} = -\frac{\lambda+2\mu}{2\pi c_2}\ln r + \frac{\lambda+\mu}{4\pi c_1 c_2}\left(\frac{1}{3r^6}[R_1x_1(x_1^2 - 3x_2^2) + R_2x_2(3x_1^2 - x_2^2)]^2 - R^2\ln r\right) \tag{31c}$$

$$G^{44} = -\frac{\lambda+2\mu}{2\pi c_2}\ln r + \frac{\lambda+\mu}{4\pi c_1 c_2}\left(\frac{1}{3r^6}[R_2x_1(x_1^2 - 3x_2^2) - R_1x_2(3x_1^2 - x_2^2)]^2 - R^2\ln r\right) \tag{31d}$$

$$G^{12} = G^{21} = \frac{(\lambda+\mu)K_1^2}{4\pi c_1 c_2}\frac{x_1x_2}{r^2} \tag{31e}$$

$$G^{13} = G^{31} = \frac{R_1}{2\pi c_2}\left(\frac{x_1^2}{r^2} + \frac{c_3}{2c_1}\frac{x_1^2(x_1^2 - x_2^2)}{r^4}\right) + \frac{R_2}{2\pi c_2}\left(\frac{x_1x_2}{r^2} + \frac{c_3}{c_1}\frac{x_1^3x_2}{r^4}\right) \tag{31f}$$

$$G^{14} = G^{41} = \frac{R_1}{2\pi c_2}\left(\frac{x_1x_2}{r^2} + \frac{c_3}{c_1}\frac{x_1^3x_2}{r^4}\right) - \frac{R_2}{2\pi c_2}\left(\frac{x_1^2}{r^2} + \frac{c_3}{2c_1}\frac{x_1^2(x_1^2 - x_2^2)}{r^4}\right) \tag{31g}$$

$$G^{23} = G^{32} = -\frac{R_1}{2\pi c_2}\left(\frac{x_1x_2}{r^2} + \frac{c_3}{c_1}\frac{x_1x_2^3}{r^4}\right) - \frac{R_2}{2\pi c_2}\left(\frac{x_2^2}{r^2} + \frac{c_3}{2c_1}\frac{x_2^2(x_2^2 - x_1^2)}{r^4}\right) \tag{31h}$$

$$G^{24} = G^{42} = -\frac{R_1}{2\pi c_2}\left(\frac{x_2^2}{r^2} + \frac{c_3}{2c_1}\frac{x_2^2(x_2^2 - x_1^2)}{r^4}\right) + \frac{R_2}{2\pi c_2}\left(\frac{x_1x_2}{r^2} + \frac{c_3}{c_1}\frac{x_1x_2^3}{r^4}\right) \tag{31i}$$

$$G^{34} = G^{43} = \frac{(\lambda+\mu)(R_1^2 - R_2^2)}{12\pi c_1 c_2}\frac{x_1x_2(3x_1^2 - x_2^2)(x_1^2 - 3x_2^2)}{r^6} - \frac{(\lambda+\mu)R_1R_2}{2\pi c_1 c_2}\left(\frac{x_1^2(x_1^2 - 3x_2^2)^2}{3r^6} + \ln r\right) \tag{31j}$$

where $R^2 = R_1^2 + R_2^2$, $c_1 = \mu K_1 - R^2$, $c_2 = (\lambda + 2\mu)K_1 - R^2$, $c_3 = K_1(\lambda + \mu)$ and $r^2 = x_1^2 + x_2^2$.

3.1.2. Planar quasicrystals of point groups 5m, 10 and 10mm. In these cases there are only five independent elastic constants, i.e. $R_2 = 0$, compared to the point group 5. Therefore, equations (27)–(31) can be used for point groups 5m, 10 and 10mm after letting $R_2 = 0$. These results agree entirely with those obtained by De *et al* [5] through another way:

$$\begin{array}{llll} G^{11} = \alpha_{11} & G^{22} = \alpha_{22} & G^{33} = \delta_{11} & G^{44} = \delta_{22} \\ G^{12} = G^{21} = \alpha_{12} = \alpha_{21} & & G^{13} = G^{31} = \gamma_{11} = \beta_{11} & \\ G^{14} = G^{41} = \gamma_{21} = \beta_{21} & & G^{23} = G^{32} = \gamma_{12} = \beta_{21} & \\ G^{24} = G^{42} = \gamma_{22} = \beta_{22} & & G^{34} = G^{43} = \delta_{12} = \delta_{21} & \end{array} \tag{32}$$

where α_{ij}, β_{ij}, γ_{ij} and δ_{ij} are symbols adopted in [5].

3.1.3. Planar quasicrystals of point group 8mm. In this case, C_{ijkl} takes the same form as equation (27), R_{ijkl} can be obtained from equation (28) letting $R_2 = 0$, and K_{ijkl} is [1]

$$K_{ijkl} = (K_1 - K_2 - K_3)\delta_{ik}\delta_{jl} + K_2\delta_{ij}\delta_{kl} + K_3\delta_{il}\delta_{jk} + 2(K_2 + K_3)(\delta_{i1}\delta_{j2}\delta_{k1}\delta_{l2} + \delta_{i2}\delta_{j1}\delta_{k2}\delta_{l1}). \tag{33}$$

It follows that the 4×4 matrix is

$$[A^{\alpha\beta}] = \begin{bmatrix} (\lambda+\mu)k_1^2 + \mu k^2 & (\lambda+\mu)k_1k_2 & R_1(k_1^2 - k_2^2) & 2R_1k_1k_2 \\ (\lambda+\mu)k_1k_2 & (\lambda+\mu)k_2^2 + \mu k^2 & -2R_1k_1k_2 & R_1(k_1^2 - k_2^2) \\ R_1(k_1^2 - k_2^2) & -2R_1k_1k_2 & K_1k^2 + (K_2 + K_3)k_2^2 & (K_2 + K_3)k_1k_2 \\ 2R_1k_1k_2 & R_1(k_1^2 - k_2^2) & (K_2 + K_3)k_1k_2 & K_1k^2 + (K_2 + K_3)k_1^2 \end{bmatrix}. \tag{34}$$

The corresponding determinant $D(\boldsymbol{k})$ and the algebraic complements $B^{\alpha\beta}(\boldsymbol{k})$ are as follows:

$$D(\boldsymbol{k}) = (a_1 - R_1^2)(a_2 - R_1^2)k^8 - 16a_4a_6k_1^2k_2^2(k_1^2 - k_2^2) \tag{35}$$

$$B^{11}(\boldsymbol{k}) = K_1(a_2 - R_1^2)k^6 + a_3K_1k^4k_2^2 - a_4R_1k_2^2(3k_1^2 - k_2^2)^2 \tag{36a}$$

$$B^{22}(\boldsymbol{k}) = K_1(a_2 - R_1^2)k^6 + a_3K_1k^4k_1^2 - a_4R_1k_1^2(k_1^2 - 3k_2^2)^2 \tag{36b}$$

$$B^{33}(\boldsymbol{k}) = \mu(a_1 - R_1^2)k^6 + \mu a_5k^4k_1^2 - a_6R_1k_1^2(k_1^2 - 3k_2^2)^2 \tag{36c}$$

$$B^{44}(\boldsymbol{k}) = \mu(a_1 - R_1^2)k^6 + \mu a_5k^4k_2^2 - a_6R_1k_2^2(3k_1^2 - k_2^2)^2 \tag{36d}$$

$$B^{12}(\mathbf{k}) = B^{21}(\mathbf{k}) = -(a_3K_1 + 3a_4R_1)k^4k_1k_2 + 16a_4R_1k_1^3k_2^3 \tag{36e}$$

$$B^{34}(\mathbf{k}) = B^{43}(\mathbf{k}) = -(a_5\mu + 3a_6R_1)k^4k_1k_2 + 16a_6R_1k_1^3k_2^3 \tag{36f}$$

$$B^{13}(\mathbf{k}) = B^{31}(\mathbf{k}) = a_8R_1k^4(k_2^2 - k_1^2) + a_6K_1k^2k_2^2(k_2^2 - 3k_1^2) + a_4\mu k^2k_1^2(3k_2^2 - k_1^2) + 4a_7R_1k_1^2k_2^2(k_2^2 - k_1^2) \tag{36g}$$

$$B^{24}(\mathbf{k}) = B^{42}(\mathbf{k}) = a_8R_1k^4(k_2^2 - k_1^2) + a_6K_1k^2k_1^2(3k_2^2 - k_1^2) + a_4\mu k^2k_2^2(k_2^2 - 3k_1^2) + 4a_7R_1k_1^2k_2^2(k_2^2 - k_1^2) \tag{36h}$$

$$B^{14}(\mathbf{k}) = B^{41}(\mathbf{k}) = -2R_1a_8k^4k_1k_2 + (\lambda K_1 + a_2)R_1k^2k_1k_2(k_1^2 - 3k_2^2) + 4a_7R_1k_1k_2^3(k_1^2 - k_2^2) \tag{36i}$$

$$B^{23}(\mathbf{k}) = B^{32}(\mathbf{k}) = 2R_1a_8k^4k_1k_2 + (\lambda K_1 + a_2)R_1k^2k_1k_2(3k_1^2 - k_2^2) + 4a_7R_1k_1^3k_2(k_1^2 - k_2^2) \tag{36j}$$

where

$$\begin{aligned} &a_1 = (\lambda + 2\mu)K_1 \qquad a_2 = (K_1 + K_2 + K_3)\mu \\ &a_3 = (\lambda + \mu)(K_1 + K_2 + K_3) \qquad a_4 = (K_2 + K_3)R_1 \\ &a_5 = (\lambda + 2\mu)(K_2 + K_3) \qquad a_6 = (\lambda + \mu)R_1 \\ &a_7 = (\lambda + \mu)(K_2 + K_3) \qquad a_8 = \mu K_1 - R_1^2. \end{aligned} \tag{37}$$

Consequently, the Fourier transforms $\bar{G}^{\alpha\beta}(\mathbf{k})$ can be obtained by equation (18) using equations (35) and (36). Unfortunately, it is difficult to write explicit expressions for $G^{\alpha\beta}(\mathbf{x})$.

3.1.4. Planar quasicrystals of point group 12mm. For this symmetry there is no interaction between the phonon and phason elastic strain fields [8, 9], so we have $R_{ijkl} = 0$. Moreover, the matrix elements C_{ijkl} and K_{ijkl} are the same as that of the planar quasicrystals with point group 8mm. Therefore, the Fourier transforms $\bar{G}^{\alpha\beta}(\mathbf{k})$ of the planar quasicrystals of point group 12mm can be obtained directly from expressions (18), (35), (36) and (37) by letting $R_1 = 0$. Carrying out the double Fourier integral (26), we have the explicit expressions for $\bar{G}^{\alpha\beta}(\mathbf{x})$ as follows:

$$G^{11}(\mathbf{x}) = -\frac{1}{4\pi}\frac{\lambda + 3\mu}{\mu(\lambda + 2\mu)}\ln r - \frac{1}{4\pi}\frac{\lambda + \mu}{\mu(\lambda + 2\mu)}\frac{x_2^2}{r^2} \tag{38a}$$

$$G^{22}(\mathbf{x}) = -\frac{1}{4\pi}\frac{\lambda + 3\mu}{\mu(\lambda + 2\mu)}\ln r - \frac{1}{4\pi}\frac{\lambda + \mu}{\mu(\lambda + 2\mu)}\frac{x_1^2}{r^2} \tag{38b}$$

$$G^{33}(\mathbf{x}) = -\frac{1}{4\pi}\frac{1}{K_1(K_1 + K_2 + K_3)}\left((2K_1 + K_2 + K_3)\ln r + (K_2 + K_3)\frac{x_1^2}{r^2}\right) \tag{38c}$$

$$G^{44}(\mathbf{x}) = -\frac{1}{4\pi}\frac{1}{K_1(K_1 + K_2 + K_3)}\left((2K_1 + K_2 + K_3)\ln r + (K_2 + K_3)\frac{x_2^2}{r^2}\right) \tag{38d}$$

$$G^{12}(\mathbf{x}) = G^{21}(\mathbf{x}) = \frac{1}{4\pi}\frac{\lambda + \mu}{\mu(\lambda + 2\mu)}\frac{x_1x_2}{r^2} \tag{38e}$$

$$G^{34}(\mathbf{x}) = G^{43}(\mathbf{x}) = \frac{1}{4\pi}\frac{K_2 + K_3}{K_1(K_1 + K_2 + K_3)}\frac{x_1x_2}{r^2} \tag{38f}$$

$$G^{13} = G^{31} = G^{14} = G^{41} = G^{23} = G^{32} = G^{24} = G^{42} = 0. \tag{38g}$$

3.2. *Quasicrystalline solids in three-dimensional space*

In this subsection, our discussions deal with three kinds of quasicrystals: icosahedral quasicrystals, 3D quasicrystals of crystalline symmetries, such as the cubic quasicrystal [10], and several 3D solids with a periodic axis and a quasiperiodic plane (2D quasicrystals). In these quasicrystals defects such as dislocations have been observed in experiment. Thus, the discussions here are of great interest both to theory and to application.

The matrices of C_{ijkl}, K_{ijkl} and R_{ijkl} of the icosahedral and cubic quasicrystals have been given in [1] and [11]. It follows that for such quasicrystals $A^{\alpha\beta}(\boldsymbol{k})$ is a 6×6 matrix. Its determinant $D(\boldsymbol{k})$ is a twelfth-order polynomial. For 3D solids with a 2D quasiperiodic plane the elastic properties have been given by us with the group representation theory [7, 12, 13]. The results show that the 9×9 matrix of C_{ijkl} $(i, j, k, l = 1, 2, 3)$ is the same as that for the hexagonal crystal, and the matrices of K_{ijkl} $(i, k = 1, 2$ and $j, l = 1, 2, 3)$ and R_{ijkl} $(i, j, l = 1, 2, 3$ and $k = 1, 2)$ are 6×6 and 9×6 matrices, respectively. Therefore, $A^{\alpha\beta}(\boldsymbol{k})$ is a 5×5 matrix and its determinant $D(\boldsymbol{k})$ is a tenth-order polynomial.

From the discussions above, we can see that it is very difficult to express the Green tensor functions $G^{\alpha\beta}(\boldsymbol{x})$ by elementary functions, and only numerical solutions can be obtained for the elastic displacement fields $\boldsymbol{u}$ and $\boldsymbol{w}$, in general.

However, it is possible still to find expressions of $\boldsymbol{u}$ and $\boldsymbol{w}$ with elementary functions under certain special conditions. Such an example will be given in the next section.

4. Application to dislocation in quasicrystals

According to the theory of defects in quasicrystals [14–16], the characteristic vector $\tilde{\boldsymbol{b}}$, i.e. the Burgers vector of a dislocation, is a direct sum

$$\tilde{\boldsymbol{b}} = \boldsymbol{b}^{\parallel} \oplus \boldsymbol{b}^{\perp} \tag{39}$$

where $\boldsymbol{b}^{\parallel}$ and $\boldsymbol{b}^{\perp}$ describe rigid translations in the phonon and phason fields, respectively. Following the Volterra procedure in quasicrystals [16], the dislocation condition is given by the following two integrals:

$$\oint_c \mathrm{d}\boldsymbol{u} = \boldsymbol{b}^{\parallel} \qquad \oint_c \mathrm{d}\boldsymbol{w} = \boldsymbol{b}^{\perp} \tag{40}$$

where c is the contour surrounding the core of the dislocation in the physical space $P_{\parallel}$.

Besides the dislocation condition, there are two questions to be answered: how to express the eigenstrains $E^*_{ij}(\boldsymbol{x}')$ and $w^*_{ij}(\boldsymbol{x}')$, and how to determine the subdomain V'.

In this section, we provide an exercise in the usage of these formulae suggested in this paper. We consider that the best way to understand general statements is to work out a specific example.

Here we consider a straight dislocation line in a three-dimensional body with decagonal symmetry: a solid that can be described as a stack of periodically spaced layers, each of which exhibits tenfold symmetry. According to the group representation theory, we have found the non-vanishing matrix elements of the elastic constants C_{ijkl}, R_{ijkl} and K_{ijkl} for this type of quasicrystal [7, 12, 13]. The results for point groups 10mm, 1022, $\overline{10}$m2 and 10/mmm, i.e. Laue class 10/mmm, show that C_{ijkl} (C_{KM}) are the same as those for hexagonal crystals (point groups 6, $\bar{6}$, 6/m, $\bar{6}$m2, 6mm, 622 and 6/mmm): $C_{11} = C_{22}$, C_{33}, C_{12}, $C_{13} = C_{23}$, $C_{44} = C_{55}$, $2C_{66} = C_{11} - C_{12}$; R_{ijkl} can be obtained from equation (28) letting $R_2 = 0$; and the expression of K_{ijkl} takes the same form as equation (29). These constant matrices are referred to the conventional quasicrystal coordinate systems: x_3 axis is parallel to the periodic direction in the physical space $P_\parallel$, and two χ_1 axes in $P_\parallel$ and $P_\perp$ are parallel to one of the 10 basic vectors in the quasiperiodic planes, respectively. It must be emphasized that the decagonal quasicrystals (3D) discussed here are not the same as the planar quasicrystals with decagonal symmetry discussed already in section 3.1.

In this paper, as an example, we only discuss a dislocation line that is parallel to the periodic direction (x_3 axis) and has an arbitrary Burgers vector $\tilde{b} = (b_1^\parallel, b_2^\parallel, b_3^\parallel, b_1^\perp, b_2^\perp)$. In this case, the dislocation coordinate system is consistent with the conventional quasicrystal coordinate system.

Because the component $b_3^\parallel$ is along the periodic direction without a corresponding component in $P_\perp$, its displacement field is the same as that in any transverse isotropic continuum, e.g. in hexagonal crystal [2]. Hence now we only consider the elastic displacement field induced by the Burgers vector $(b_1^\parallel, b_2^\parallel, 0, b_1^\perp, b_2^\perp)$. In analogy to the work of De *et al* [5], first we calculate the field induced by $(b_1^\parallel, 0, 0, b_1^\perp, 0)$. The results for $(0, b_2^\parallel, 0, 0, b_2^\perp)$ can then be obtained from the expressions for $(b_1^\parallel, 0, 0, b_1^\perp, 0)$ by rotating the coordinate system in the parallel space $P_\parallel$ by $\pi/2$ and that in the perpendicular space $P_\perp$ by $3\pi/2$ anticlockwise, respectively. It is easy to prove that the matrix forms of the elastic constants C_{ijkl}, R_{ijkl} and K_{ijkl} remain unchanged after such a coordinate transformation.

In the case of a dislocation line parallel to the x_3 axis, the displacement vector $V^\alpha(x_1, x_2, x_3)$ is independent of the coordinate x_3: $V^\alpha = V^\alpha(x_1, x_2)$.

For $(b_1^\parallel, 0, 0, b_1^\perp, 0)$, the subdomain V' consists of a semi-infinite plane $(x_1' < 0, x_2' = 0, -\infty \leqslant x_3' \leqslant +\infty)$ in $P_\parallel$. The eigenstrains can be given as follows:

$$E_{12}^* = E_{21}^* = \tfrac{1}{2} b_1^\parallel \delta(x_2') H(-x_1') \tag{41}$$

$$w_{12}^* = b_1^\perp \delta(x_2') H(-x_1') \tag{42}$$

where $\delta(x)$ is the Dirac delta function and $H(-x)$ is the Heaviside step function.

Substituting equations (22), (41) and (42) into (25), we have

$$\begin{aligned} V^\alpha(x) = {} & \frac{b_1^\parallel}{4\pi^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\delta_i^\beta C_{ij12} + \delta_i^{\beta-3} R_{12ij}) \left(\frac{k_j}{k_1} \bar{G}^{\alpha\beta}(k)\right)_{k_3=0} \exp[\mathrm{i}(k_1x_1 + k_2x_2)]\,\mathrm{d}k_1\,\mathrm{d}k_2 \\ & + \frac{b_1^\perp}{4\pi^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\delta_i^\beta C_{ij12} + \delta_i^{\beta-3} K_{ij12}) \left(\frac{k_j}{k_1} \bar{G}^{\alpha\beta}(k)\right)_{k_3=0} \\ & \times \exp[\mathrm{i}(k_1x_1 + k_2x_2)]\,\mathrm{d}k_1\,\mathrm{d}k_2. \end{aligned} \tag{43}$$

Next, we calculate the Fourier transforms $\bar{G}^{\alpha\beta}(k)$ of the Green functions. Fortunately, when $k_3 = 0$, $\bar{G}^{\alpha\beta}(k)$ for decagonal quasicrystals with point groups 10/mmm, 10mm, 1022 and $\overline{10}$m2 can be expressed by elementary functions. The 5×5 matrix $A^{\alpha\beta}(k)$ is given as follows:

$$[A^{\alpha\beta}] = \begin{bmatrix} C_{11}k_1^2 + C_{66}k_2^2 & (C_{11} - C_{66})k_1k_2 & 0 & R_1(k_1^2 - k_2^2) & 2R_1k_1k_2 \\ (C_{11} - C_{66})k_1k_2 & C_{66}k_1^2 + C_{11}k_2^2 & 0 & -2R_1k_1k_2 & R_1(k_1^2 - k_2^2) \\ 0 & 0 & C_{44}k^2 & 0 & 0 \\ R_1(k_1^2 - k_2^2) & -2R_1k_1k_2 & 0 & K_1k^2 & 0 \\ 2R_1k_1k_2 & R_1(k_1^2 - k_2^2) & 0 & & K_1k^2 \end{bmatrix}. \tag{44}$$

After calculating its determinant $D(k)$ and the algebraic complements $B^{\alpha\beta}(k)$, we obtain the Fourier transformations $\bar{G}^{\alpha\beta}(k)$ when $k_3 = 0$:

$$\bar{G}^{11} = -\frac{K_1 R_1^2}{C}\frac{1}{k^2} + \frac{K_1^2}{C}\frac{C_{66}k_1^2 + C_{11}k_2^2}{k^4} \tag{45a}$$

$$\bar{G}^{22} = -\frac{K_1 R_1^2}{C}\frac{1}{k^2} + \frac{K_1^2}{C}\frac{C_{11}k_1^2 + C_{66}k_2^2}{k^4} \tag{45b}$$

$$\bar{G}^{33} = \frac{1}{C_{44}}\frac{1}{k^2} \tag{45c}$$

$$\bar{G}^{44} = \frac{C_{11}C_{66}K_1}{C}\frac{1}{k^2} - \frac{R_1^2}{C}\frac{C_{11}k_1^2 + C_{66}k_2^2}{k^4} + \frac{8(C_{11} - C_{66})R_1^2}{C}\frac{(k_1^2 - k_2^2)k_1^2 k_2^2}{k^8} \tag{45d}$$

$$\bar{G}^{55} = \frac{C_{11}C_{66}K_1}{C}\frac{1}{k^2} - \frac{R_1^2}{C}\frac{C_{66}k_1^2 + C_{11}k_2^2}{k^4} - \frac{8(C_{11} - C_{66})R_1^2}{C}\frac{(k_1^2 - k_2^2)k_1^2 k_2^2}{k^8} \tag{45e}$$

$$\bar{G}^{12} = \bar{G}^{21} = -\frac{(C_{11} - C_{66})K_1^2}{C}\frac{k_1 k_2}{k^4} \tag{45f}$$

$$\bar{G}^{14} = \bar{G}^{41} = \frac{R_1(R_1^2 - C_{66}K_1)}{C}\frac{1}{k^2} - \frac{2R_1^3 + K_1 R_1(3C_{11} - 5C_{66})}{C}\frac{k_2^2}{k^4} + \frac{4K_1 R_1(C_{11} - C_{66})}{C}\frac{k_2^4}{k^6} \tag{45g}$$

$$\bar{G}^{15} = \bar{G}^{51} = \frac{2R_1^3 + K_1 R_1(C_{11} - 3C_{66})}{C}\frac{k_1 k_2}{k^4} - \frac{4K_1 R_1(C_{11} - C_{66})}{C}\frac{k_1 k_2^3}{k^6} \tag{45h}$$

$$\bar{G}^{24} = \bar{G}^{42} = \frac{K_1 R_1(3C_{11} - C_{66}) - 2R_1^3}{C}\frac{k_1 k_2}{k^4} - \frac{4K_1 R_1(C_{11} - C_{66})}{C}\frac{k_1 k_2^3}{k^6} \tag{45i}$$

$$\bar{G}^{25} = \bar{G}^{52} = \frac{R_1(C_{66}K_1 - R_1^2)}{C}\frac{1}{k^2} + \frac{2R_1^3 + K_1 R_1(3C_{11} - 5C_{66})}{C}\frac{k_1^2}{k^4} - \frac{4K_1 R_1(C_{11} - C_{66})}{C}\frac{k_1^4}{k^6} \tag{45j}$$

$$\bar{G}^{45} = \bar{G}^{54} = -\frac{3(C_{11} - C_{66})R_1^2}{C}\frac{k_1 k_2}{k^4} + \frac{16R_1^2(C_{11} - C_{66})}{C}\frac{k_1^3 k_2^3}{k^8} \tag{45k}$$

$$\bar{G}^{13} = \bar{G}^{31} = \bar{G}^{23} = \bar{G}^{32} = \bar{G}^{34} = \bar{G}^{43} = \bar{G}^{35} = \bar{G}^{53} = 0 \tag{45l}$$

where

$$k^2 = k_1^2 + k_2^2 \qquad C = (C_{66}K_1 - R_1^2)(C_{11}K_1 - R_1^2). \tag{46}$$

Substituting equation (45) into equation (43) and carrying out the Fourier integrals, we obtain the expressions of the displacement field for $(b_1^{\parallel}, 0, 0, b_1^{\perp}, 0)$. Then, completing the coordinate transformation mentioned above, the expressions corresponding to Burgers vector $(0, b_2^{\parallel}, 0, 0, b_2^{\perp})$ can be found.

The final results are as follows: the phonon displacement field,

$$u_1(x_1, x_2) = \frac{b_1^{\parallel}}{2\pi}\left[\tan^{-1}\left(\frac{x_2}{x_1}\right) + \frac{K_1(C_{11} - C_{66})}{C_{11}K_1 - R_1^2}\frac{x_1 x_2}{r^2}\right]$$

$$
\begin{aligned}
&+\frac{b_2^{\parallel}}{2\pi}\left[\frac{C_{66}K_1-R_1^2}{C_{11}K_1-R_1^2}\ln\left(\frac{r}{r_0}\right)-\frac{K_1(C_{11}-C_{66})}{C_{11}K_1-R_1^2}\frac{x_1^2}{r^2}\right]\\
&+\frac{b_1^{\perp}}{2\pi}\left(\frac{R_1(K_1-K_2)}{C_{66}K_1-R_1^2}\frac{x_1x_2}{r^2}-\frac{K_1R_1(K_1-K_2)(C_{11}-C_{66})}{(C_{66}K_1-R_1^2)(C_{11}K_1-R_1^2)}\frac{x_1x_2^3}{r^4}\right)\\
&-\frac{b_2^{\perp}}{2\pi}\left(\frac{R_1(K_1-K_2)}{C_{11}K_1-R_1^2}\frac{x_1^2}{r^2}+\frac{R_1K_1(K_1-K_2)(C_{11}-C_{66})}{2(C_{66}K_1-R_1^2)(C_{11}K_1-R_1^2)}\frac{x_1^2(x_1^2-x_2^2)}{r^4}\right)
\end{aligned}
\tag{47a}
$$

$$
\begin{aligned}
u_2(x_1,x_2)=&\frac{b_1^{\parallel}}{2\pi}\left[\frac{R_1^2-C_{66}K_1}{C_{11}K_1-R_1^2}\ln\left(\frac{r}{r_0}\right)+\frac{K_1(C_{11}-C_{66})}{C_{11}K_1-R_1^2}\frac{x_2^2}{r^2}\right]\\
&+\frac{b_2^{\parallel}}{2\pi}\left[\tan^{-1}\left(\frac{x_2}{x_1}\right)-\frac{K_1(C_{11}-C_{66})}{C_{11}K_1-R_1^2}\frac{x_1x_2}{r^2}\right]\\
&+\frac{b_1^{\perp}}{2\pi}\left(-\frac{R_1(K_1-K_2)}{C_{11}K_1-R_1^2}\frac{x_2^2}{r^2}+\frac{K_1R_1(K_1-K_2)(C_{11}-C_{66})}{2(C_{66}K_1-R_1^2)(C_{11}K_1-R_1^2)}\frac{x_2^2(x_1^2-x_2^2)}{r^4}\right)\\
&+\frac{b_2^{\perp}}{2\pi}\left(\frac{R_1(K_1-K_2)}{C_{66}K_1-R_1^2}\frac{x_1x_2}{r^2}-\frac{K_1R_1(K_1-K_2)(C_{11}-C_{66})}{(C_{66}K_1-R_1^2)(C_{11}K_1-R_1^2)}\frac{x_1^3x_2}{r^4}\right)
\end{aligned}
\tag{47b}
$$

$$
u_3(x_1,x_2)=0 \tag{47c}
$$

and the phason displacement field,

$$
\begin{aligned}
w_1(x_1,x_2)=&\frac{b_1^{\parallel}}{2\pi}\left[\frac{2R_1(C_{11}-C_{66})}{C_{11}K_1-R_1^2}\left(\frac{x_1x_2}{r^2}-\frac{x_1x_2^3}{r^4}\right)\right]\\
&+\frac{b_2^{\parallel}}{2\pi}\frac{R_1(C_{11}-C_{66})}{C_{11}K_1-R_1^2}\left(\frac{2x_1^2}{r^2}-\frac{x_1^2(x_1^2-x_2^2)}{r^4}\right)\\
&+\frac{b_1^{\perp}}{2\pi}\left[\tan^{-1}\left(\frac{x_2}{x_1}\right)+\frac{R_1^2(K_1-K_2)(C_{11}-C_{66})}{(C_{66}K_1-R_1^2)(C_{11}K_1-R_1^2)}\left(\frac{x_1x_2}{2r^2}-\frac{8x_1^3x_2^3}{3r^6}\right)\right]\\
&+\frac{b_2^{\perp}}{2\pi}\left[\frac{R_1^2[(K_1+K_2)(C_{11}-C_{66})-2R_1^2]}{2(C_{66}K_1-R_1^2)(C_{11}K_1-R_1^2)}\ln\left(\frac{r}{r_0}\right)\right.\\
&\left.-\frac{R_1^2(K_1-K_2)(C_{11}-C_{66})}{(C_{66}K_1-R_1^2)(C_{11}K_1-R_1^2)}\frac{x_1^2(x_1^2-3x_2^2)^2}{6r^6}\right]
\end{aligned}
\tag{47d}
$$

$$
\begin{aligned}
w_2(x_1,x_2)=&\frac{b_1^{\parallel}}{2\pi}\frac{R_1(C_{11}-C_{66})}{C_{11}K_1-R_1^2}\left(\frac{2x_2^2}{r^2}+\frac{x_2^2(x_1^2-x_2^2)}{r^4}\right)\\
&+\frac{b_2^{\parallel}}{2\pi}\frac{2R_1(C_{11}-C_{66})}{C_{11}K_1-R_1^2}\left(\frac{x_1x_2}{r^2}-\frac{x_1^3x_2}{r^4}\right)\\
&+\frac{b_1^{\perp}}{2\pi}\left[\frac{R_1^2(K_1-K_2)(C_{11}-C_{66})}{(C_{66}K_1-R_1^2)(C_{11}K_1-R_1^2)}\frac{x_1^2(3x_1^2-x_2^2)^2}{6r^6}\right.\\
&\left.+\frac{R_1^2[2R_1^2-(K_1+K_2)(C_{11}+C_{66})]}{2(C_{66}K_1-R_1^2)(C_{11}K_1-R_1^2)}\ln\left(\frac{r}{r_0}\right)\right]
\end{aligned}
$$

$$+\frac{b_2^{\perp}}{2\pi}\left[\tan^{-1}\left(\frac{x_2}{x_1}\right)-\frac{R_1^2(K_1-K_2)(C_{11}-C_{66})}{(C_{66}K_1-R_1^2)(C_{11}K_1-R_1^2)}\left(\frac{x_1x_2}{2r^2}-\frac{8x_1^3x_2^3}{3r^6}\right)\right] \tag{47e}$$

where $r^2 = x_1^2 + x_2^2$ and r_0 is the radius of dislocation core.

5. Discussion

Now we discuss the range suitable for the application of the elastic model derived in this paper. The elastic model of dislocations is based on the generalized elasticity theory of quasicrystals [1] in which the elastic free energy is described as proportional to the square gradient of phason variables. It is well known, for describing quasicrystal structures, that there are two basic approaches: the continuum density-wave picture and the unit-cell picture. Both the pictures are useful for studying the physical properties of quasicrystals. Nevertheless, as has already been pointed out in [18], there are fundamental open questions. In the density-wave picture the elasticity energy grows as the square of the gradient in the phason field, just as for phonons. In the unit-cell picture, phason excitations correspond to local discrete rearrangements. If one associates some fixed energy per mismatch in the unit-cell picture, one finds that the energy scales linearly rather than quadratically with the gradient in $\boldsymbol{w}$. If the unit-cell picture is correct, what is the complete elasticity theory of quasicrystals? Recently, Jeong and Steinhardt [19] have studied the phason dynamics of an energetically stabilized tiling model for three-dimensional bodies with decagonal symmetry. According to their results, there is a transition temperature T_c. When $T < T_c$ the system is in the locked phase, in which the energy is proportional to $|\Delta \boldsymbol{w}|$. When $T > T_c$ the system is in the unlocked phase and the elastic energy grows as $(\Delta \boldsymbol{w})^2$. It follows that the elastic model discussed in the present paper is correct for quasicrystals in the unlocked phase.

As has already been pointed out in section 4 the elasticity property of decagonal quasicrystals (3D bodies) is different from that of planar decagonal quasicrystals (2D). Hence, the expressions of the elastic fields of dislocations in both of them are different from each other, in general. However, it is possible to have the same form in some special case. For example, as discussed in section 4, because the dislocation line is along the periodic axis (x_3) of the decagonal quasicrystal (3D), we have all $\partial_3 = 0$ for $\boldsymbol{u}$ and $\boldsymbol{w}$. In consequence, the expressions (47*a*)–(47*e*) in the present paper will be consistent with expressions (4.9*a*)–(4.9*d*) in [5] obtained by De *et al* for a dislocation (point) in a planar quasicrystal provided we take $(\lambda+2\mu)$ and μ instead of C_{11} and C_{66}, respectively. But when a straight dislocation line is lying on the decagonal symmetry plane this is not the case. For this dislocation line, the expression of the elastic displacement and the stress fields have been derived using a generalized Eshelby's method developed by us [20].

By using the general expression (25), recently, we have calculated the elastic displacement fields of dislocation lines either parallel to the periodic direction or lying on the quasiperiodic plane of a dodecagonal quasicrystal (3D solid) [21]. Moreover, besides the expressions of $\boldsymbol{u}$ and $\boldsymbol{w}$, we have also given 2D tiling pictures, which show the effects of the phason and the total displacement fields induced by a dislocation line. These pictures illustrate that the results derived from this elastic model are in agreement with the fundamental theory of dislocations in quasicrystals [16].

By using the defocused convergent-beam electron diffraction technique, Burgers vectors of dislocations in Al–Co–Ni decagonal quasicrystal were identified [17] to be $\tilde{b} = \langle 0, 1, 0, 0, -1, 0\rangle$ and $\tilde{b} = \langle 0, 1, 1, -1, -1, 0\rangle$, which correspond to $\tilde{b} = (0, b_2^{\parallel}, 0, 0, -b_2^{\perp})$

in the coordinate system used here with $|-b^{\perp}|/|b^{\parallel}| = 1/\tau$ for the former and $|-b^{\perp}|/|b^{\parallel}| = 1/\tau^3$ for the latter ($\tau = (1+\sqrt{5})/2$). When the dislocation lines are along the periodic direction, their displacement fields can be obtained from equation (47) after substituting concrete values in these equations.

Notice that the dislocation coordinate system in this specific example is the same as the quasicrystal coordinate system, which simplified the calculation. Otherwise a coordinate transformation must be carried out accordingly.

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